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# Surface-Only Dynamic Deformables using a Boundary Element Method - Supplementary Material

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## B. Derivation of surface-only body force terms

We use a non-bold font with subscripts to denote components of a vector or matrix in this note. The time domain displacement fundamental solution for elastodynamics [EŞ74] $\mathbf{u}^*$ is given by

$$u^*_{ij}(\mathbf{x},\mathbf{y},t') = \frac{1}{4\pi\rho}\left\{\frac{t'}{r^2}\left(\frac{3r_ir_j}{r^3}-\frac{\delta_{ij}}{r}\right)\left[H\left(t'-\frac{r}{c_1}\right)-H\left(t'-\frac{r}{c_2}\right)\right]\right.$$
$$\left.+\frac{r_ir_j}{r^3}\left[\frac{1}{c_1^2}\delta\left(t'-\frac{r}{c_1}\right)-\frac{1}{c_2^2}\delta\left(t'-\frac{r}{c_2}\right)\right]+\frac{\delta_{ij}}{rc_2^2}\delta\left(t'-\frac{r}{c_2}\right)\right\}, \tag{22}$$

where $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $r=\sqrt{r_1^2+r_2^2+r_3^2}$, $\mathbf{n}(\mathbf{y})$ is the outward unit normal at $\mathbf{y}$, $\delta_{ij}$ is Kronecker's delta, $H(\cdot)$ is the Heaviside step function, and $c_1$ and $c_2$ are the longitudinal and shear wave speeds, respectively, computed from the material parameters.

We will convert volume integrals involving time domain fundamental solutions into surface integrals as outlined in Eq. (4), and apply the Laplace transform in the end to get the Laplace domain functions for body force terms. To describe the process, we use the following notation (Fig. 10):

- $\mathbf{r}=\mathbf{y}-\mathbf{x}$,
- $t'=t-\tau$,
- $\Omega_r$ : Subset of the domain $\Omega$ where $r<c_\eta t'$,
- $\Gamma_r$ : Curved surface inside the domain $\Omega$ where $r=c_\eta t'$, and
- $\Gamma_+$ : Subset of the boundary $\Gamma$ where $r<c_\eta t'$,

where $\eta=1,2$. The domains $\Omega_r$, $\Gamma_r$ and $\Gamma_+$ depend on $c_\eta$ and $t'$, but we omit the explicit dependencies notation for the sake of brevity.

### B.1. Translational fictitious force and gravitational force

We substitute the body force due to the translational fictitious force and gravitational force $\mathbf{b}(\mathbf{y},\tau)=\rho\mathbf{g}'(\tau)-\rho\mathbf{a}(\tau)$ into the body force term in the boundary integral equation:

$$\begin{aligned}&\int_0^t\int_\Omega \mathbf{u}^*(\mathbf{x},\mathbf{y},t-\tau)\mathbf{b}(\mathbf{y},\tau)\,d\Omega_\mathbf{y}\,d\tau\\ =&\int_0^t\left(\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y},t-\tau)\,d\Omega_\mathbf{y}\right)(\mathbf{a}(\tau)-\mathbf{g}'(\tau))\,d\tau.\end{aligned} \tag{23}$$

We will convert the volume integral in the last line of Eq. (23) to a surface integral.

Observe that, omitting the constants, there are three different types of terms in the integrand of the volume integral, $\mathbf{u}^*$ (Eq. (22)):

i $\left(\frac{3r_ir_j}{r^5}-\frac{\delta_{ij}}{r^3}\right)H\left(t'-\frac{r}{c_\eta}\right)$ $(\eta=1,2)$,
ii $\frac{r_ir_j}{r^3}\delta\left(t'-\frac{r}{c_\eta}\right)$, and
iii $\frac{1}{r}\delta\left(t'-\frac{r}{c_2}\right)$.

We will separately consider how each of them can be converted to a surface integral.

(i) The terms of the first type are converted to surface integrals using the divergence theorem:

$$\begin{aligned}&\int_\Omega\left(\frac{3r_ir_j}{r^5}-\frac{\delta_{ij}}{r^3}\right)H\left(t'-\frac{r}{c_\eta}\right)d\Omega_\mathbf{y}\\ =&\int_{\Omega_r}\left(\frac{3r_ir_j}{r^5}-\frac{\delta_{ij}}{r^3}\right)d\Omega_\mathbf{y}\\ =&-\int_{\Omega_r}\nabla\cdot\frac{r_i}{r^3}\mathbf{e}_j\,d\Omega_\mathbf{y}\\ =&-\int_{\Gamma_+\cup\Gamma_r}\frac{r_in_j}{r^3}\,d\Gamma_\mathbf{y}\quad(\because \text{divergence theorem})\\ =&-\int_{\Gamma_+}\frac{r_in_j}{r^3}\,d\Gamma_\mathbf{y}-\int_{\Gamma_r}\frac{r_in_j}{r^3}\,d\Gamma_\mathbf{y}\\ =&-\int_{\Gamma}\frac{r_in_j}{r^3}H\left(t'-\frac{r}{c_\eta}\right)d\Gamma_\mathbf{y}-\int_{\Gamma_r}\frac{r_in_j}{r^3}\,d\Gamma_\mathbf{y},\end{aligned} \tag{24}$$

where $\mathbf{e}_j$ is the $j^{th}$ standard basis vector. Notice that the first term is an integral over the boundary of the volumetric domain, but the second term is an integral over $\Gamma_r$, the wave front of the wave with wave speed $c_\eta$. We will show next that this term is cancelled out.

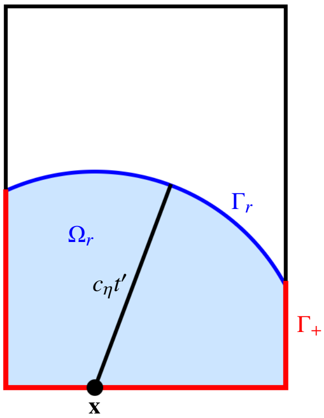


**Figure 10:** *2D illustration of notation. The rectangle represents the domain of object* $\Omega$.

(ii) The terms of the second type are converted to surface integrals using a property of the Dirac delta function:

$$\begin{aligned}&\int_\Omega \frac{r_i r_j}{r^3}\delta\left(t'-\frac{r}{c_\eta}\right)d\Omega_\mathbf{y}\\ =&c_\eta \int_\Omega \frac{r_i r_j}{r^3}\delta(r-c_\eta t')\,d\Omega_\mathbf{y} \quad \left(\because \left|\nabla\left(t'-\frac{r}{c_\eta}\right)\right|^{-1}=c_\eta\right)\\ =&c_\eta \int_{\Gamma_r}\frac{r_i r_j}{r^3}\,d\Gamma_\mathbf{y}\\ =&c_\eta^2 t'\int_{\Gamma_r}\frac{r_i n_j}{r^3}\,d\Gamma_\mathbf{y}.\end{aligned} \tag{25}$$

Observe that this cancels out the second term in Eq. (24) when we put the constants back.

(iii) The term of the last type is converted to a surface integral using the divergence theorem and Green's function for the Laplace equation:

$$\begin{aligned}&\int_\Omega \frac{1}{r}\delta\left(t'-\frac{r}{c_2}\right)d\Omega_\mathbf{y}\\ =&c_2\int_{\Gamma_r}\frac{1}{r}\,d\Gamma_\mathbf{y} \quad \left(\because\left|\nabla\left(t'-\frac{r}{c_2}\right)\right|^{-1}=c_2\right)\\ =&\frac{1}{c_2 t'^2}\int_{\Gamma_r}\mathbf{r}\cdot\mathbf{n}\,d\Gamma_\mathbf{y}\\ =&\frac{1}{c_2 t'^2}\int_{\Omega_r}\nabla\cdot\mathbf{r}\,d\Omega_\mathbf{y}-\frac{1}{c_2 t'^2}\int_{\Gamma_+}\mathbf{r}\cdot\mathbf{n}\,d\Gamma_\mathbf{y}\\ =&\frac{3}{c_2 t'^2}\int_{\Omega_r}d\Omega_\mathbf{y}-\frac{1}{c_2 t'^2}\int_\Gamma \mathbf{r}\cdot\mathbf{n}\,H\left(t'-\frac{r}{c_2}\right)d\Gamma_\mathbf{y}.\end{aligned} \tag{26}$$

Note that the first term is a volume of $\Omega_r$, which is an intersection of the original volume and a ball with radius $c_2 t'$ and center $\mathbf{x}$. Let the domain of this ball be $\Omega_{ball}$ and $\Gamma_{ball}=\partial\Omega_{ball}$.

Then, this volume integral term is computed as follows:

$$\begin{aligned}&\int_{\Omega_r}d\Omega_\mathbf{y}\\ =&\int_\Omega\int_{\Omega_{ball}}\delta(\mathbf{y}-\mathbf{z})\,d\Omega_\mathbf{z}\,d\Omega_\mathbf{y}\\ =&\int_\Gamma\left(\int_{\Gamma_{ball}}\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\mathbf{n}(\mathbf{z})\,d\Gamma_\mathbf{z}\right)\cdot\mathbf{n}(\mathbf{y})\,d\Gamma_\mathbf{y} \quad (\because \text{[Use14]})\\ =&\int_\Gamma\left(\int_{\Gamma_{ball}}\frac{\mathbf{z}}{4\pi z|\mathbf{y}-\mathbf{z}|}\,d\Gamma_\mathbf{z}\right)\cdot\mathbf{n}(\mathbf{y})\,d\Gamma_\mathbf{y}\\ =&\int_\Gamma\left[\left(\frac{1}{3}-\frac{c_2^3 t'^3}{3r^3}\right)H\left(t'-\frac{r}{c_2}\right)+\frac{c_2^3 t'^3}{3r^3}\right]\mathbf{r}\cdot\mathbf{n}(\mathbf{y})\,d\Gamma_\mathbf{y}.\end{aligned} \tag{27}$$

Therefore,

$$\int_\Omega \frac{1}{r}\delta\left(t'-\frac{r}{c_2}\right)d\Omega_\mathbf{y}=c_2^2 t'\int_\Gamma\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\left[1-H\left(t'-\frac{r}{c_2}\right)\right]d\Gamma_\mathbf{y}. \tag{28}$$

Due to (i) through (iii), we find $\mathbf{d}^*$ such that $\int_\Omega -\rho\mathbf{u}^*\,d\Omega_\mathbf{y}=\int_\Gamma \mathbf{d}^*\,d\Gamma_\mathbf{y}$:

$$\begin{aligned}d_{ij}^*(\mathbf{x},\mathbf{y},t')=-\frac{1}{4\pi}\Bigg\{&\frac{r_i n_j}{r^3}t'\left[H\left(t'-\frac{r}{c_2}\right)-H\left(t'-\frac{r}{c_1}\right)\right]\\ &+\frac{\delta_{ij}\,\mathbf{r}\cdot\mathbf{n}}{r^3}t\left[1-H\left(t'-\frac{r}{c_2}\right)\right]\Bigg\}.\end{aligned} \tag{29}$$

Its Laplace transform is as follows:

$$\begin{aligned}\hat{d}_{ij}^*(\mathbf{x},\mathbf{y},s)=-\frac{1}{4\pi}\Bigg\{&\frac{r_i n_j}{r^3 s^2}\left[\left(s\frac{r}{c_2}+1\right)e^{-\frac{r}{c_2}s}-\left(s\frac{r}{c_1}+1\right)e^{-\frac{r}{c_1}s}\right]\\ &+\frac{\delta_{ij}\,\mathbf{r}\cdot\mathbf{n}}{r^3 s^2}\left[1-\left(s\frac{r}{c_2}+1\right)e^{-\frac{r}{c_2}s}\right]\Bigg\}.\end{aligned} \tag{30}$$

### B.2. Euler force

We substitute the body force due to the Euler force, given by $\mathbf{b}(\mathbf{y},\tau)=-\rho[\mathbf{y}-\bar{\mathbf{x}}]_\times^T\alpha(\tau)$ where $[\cdot]_\times$ denotes a skew-symmetric cross product matrix, into the body force term in the boundary integral equation:

$$\begin{aligned}&\int_0^t\int_\Omega \mathbf{u}^*(\mathbf{x},\mathbf{y},t-\tau)\mathbf{b}(\mathbf{y},\tau)\,d\Omega_\mathbf{y}\,d\tau\\ =&\int_0^t\left(\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y},t-\tau)[\mathbf{y}-\bar{\mathbf{x}}]_\times^T\,d\Omega_\mathbf{y}\right)\alpha(\tau)\,d\tau\end{aligned} \tag{31}$$

The expression $\int_\Omega -\rho\mathbf{u}^*[\mathbf{y}-\bar{\mathbf{x}}]_\times^T\,d\Omega$ can be decomposed into two terms, and one of them can be converted to a boundary integral using $\mathbf{d}^*$ (Eq. (29)) we derived for the translational fictitious force and gravitational force:

$$\begin{aligned}&\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y})[\mathbf{y}-\bar{\mathbf{x}}]_\times^T\,d\Omega_\mathbf{y}\\ =&\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y})[\mathbf{y}-\mathbf{x}]_\times^T\,d\Omega_\mathbf{y}+\left(\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y})\,d\Omega_\mathbf{y}\right)[\mathbf{x}-\bar{\mathbf{x}}]_\times^T\\ =&\int_\Omega -\rho\mathbf{u}^*(\mathbf{x},\mathbf{y})[\mathbf{y}-\mathbf{x}]_\times^T\,d\Omega_\mathbf{y}+\left(\int_\Gamma \mathbf{d}^*(\mathbf{x},\mathbf{y})\,d\Gamma_\mathbf{y}\right)[\mathbf{x}-\bar{\mathbf{x}}]_\times^T.\end{aligned} \tag{32}$$

We dropped the time variables in the equation above for simplicity. We will next convert the first term to a boundary integral. Each element of $\mathbf{u}^*[\mathbf{r}]_\times^T$ consists of multiplications of an element of $\mathbf{u}^*$ and an element of $\mathbf{r}$:

$$(\mathbf{u}^*[\mathbf{r}]_\times^T)_{ij} = u^*_{i(j+2)} r_{j+1} - u^*_{i(j+1)} r_{j+2}. \tag{33}$$

We observe that $\mathbf{u}^*[\mathbf{r}]_\times^T$, which consists of $u^*_{ij} r_k (j \neq k)$, has three different types of terms, and we convert the domain integral of each term to a boundary integral:

iv $\left(\frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right) r_k H\left(t' - \frac{r}{c_\eta}\right) \quad (\eta = 1, 2)$,
v $\frac{r_i r_j r_k}{r^3} \delta\left(t' - \frac{r}{c_\eta}\right)$, and
vi $\frac{r_k}{r} \delta\left(t' - \frac{r}{c_2}\right)$.

(iv) The terms of the first type are converted to surface integrals using the divergence theorem:

$$\begin{aligned}
&\int_\Omega \left(\frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right) r_k H\left(t' - \frac{r}{c_\eta}\right) d\Omega_\mathbf{y} \\
=& \int_{\Omega_r} \left(\frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right) r_k \, d\Omega_\mathbf{y} \\
=& -\int_{\Omega_r} \left(\nabla \cdot \frac{r_i}{r^3} \mathbf{e}_j\right) r_k \, d\Omega_\mathbf{y} \\
=& -\int_{\Gamma_+ \cup \Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y} + \int_{\Omega_r} \nabla r_k \cdot \left(\frac{r_i}{r^3} \mathbf{e}_j\right) d\Omega_\mathbf{y} \\
=& -\int_{\Gamma_+ \cup \Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y} + \delta_{jk} \int_{\Omega_r} \frac{r_i}{r^3} \, d\Omega_\mathbf{y} \\
=& -\int_{\Gamma_+ \cup \Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y} \quad (\because j \neq k) \\
=& -\int_{\Gamma_+} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y} - \int_{\Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y} \\
=& -\int_\Gamma \frac{r_i r_k n_j}{r^3} H\left(t' - \frac{r}{c_\eta}\right) d\Gamma_\mathbf{y} - \int_{\Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y}.
\end{aligned} \tag{34}$$

(v) The terms of the second type are converted to surface integrals using a property of the Dirac delta function:

$$\begin{aligned}
&\int_\Omega \frac{r_i r_j r_k}{r^3} \delta\left(t' - \frac{r}{c_\eta}\right) d\Omega_\mathbf{y} \\
=& c_\eta \int_{\Gamma_r} \frac{r_i r_j r_k}{r^3} \, d\Gamma_\mathbf{y} \quad \left(\because \left|\nabla\left(t' - \frac{r}{c_\eta}\right)\right|^{-1} = c_\eta\right) \\
=& c_\eta^2 t' \int_{\Gamma_r} \frac{r_i r_k n_j}{r^3} \, d\Gamma_\mathbf{y}.
\end{aligned} \tag{35}$$

Observe that this cancels out the second term in Eq. (34) when we put the constants back.

(vi) The terms of the last type are converted to surface integrals using the divergence theorem:

$$\begin{aligned}
&\int_\Omega \frac{r_k}{r} \delta\left(t' - \frac{r}{c_2}\right) d\Omega_\mathbf{y} \\
=& c_2 \int_{\Gamma_r} \frac{r_k}{r} \, d\Gamma_\mathbf{y} \quad \left(\because \left|\nabla\left(t' - \frac{r}{c_2}\right)\right|^{-1} = c_2\right) \\
=& c_2 \int_{\Gamma_r} \mathbf{e}_k \cdot \mathbf{n} \, d\Gamma_\mathbf{y} \\
=& c_2 \int_{\Omega_r} \nabla \cdot \mathbf{e}_k \, d\Omega_\mathbf{y} - c_2 \int_{\Gamma_+} \mathbf{e}_k \cdot \mathbf{n} \, d\Gamma_\mathbf{y} \\
=& -c_2 \int_\Gamma n_k \, H\left(t' - \frac{r}{c_2}\right) d\Gamma_\mathbf{y}.
\end{aligned} \tag{36}$$

Due to (iv) through (vi), we find $\mathbf{q}^*$ such that $\int_\Omega -\rho \mathbf{u}^*[\mathbf{r}]_\times^T \, d\Omega_\mathbf{y} = \int_\Gamma \mathbf{q}^* \, d\Gamma_\mathbf{y}$:

$$\mathbf{q}^* = \mathbf{l}^* + \mathbf{d}^*[\mathbf{x} - \bar{\mathbf{x}}]_\times^T, \tag{37}$$

where $\mathbf{l}^*$ is

$$\begin{aligned}
&l^*_{ij}(\mathbf{x}, \mathbf{y}, t') \\
=& -\frac{1}{4\pi} \left\{ \frac{r_i r_{j+1} n_{j+2} - r_i r_{j+2} n_{j+1}}{r^3} t \left[ H\left(t' - \frac{r}{c_2}\right) - H\left(t' - \frac{r}{c_1}\right) \right] \right. \\
&\left. + \frac{\delta_{i(j+1)} n_{j+2} - \delta_{i(j+2)} n_{j+1}}{c_2} H\left(t' - \frac{r}{c_2}\right) \right\}.
\end{aligned} \tag{38}$$

Its Laplace transform is

$$\begin{aligned}
&\hat{l}^*_{ij}(\mathbf{x}, \mathbf{y}, s) \\
=& -\frac{1}{4\pi} \left\{ \frac{r_i r_{j+1} n_{j+2} - r_i r_{j+2} n_{j+1}}{r^3 s^2} \left[ \left(s\frac{r}{c_2} + 1\right) e^{-\frac{r}{c_2} s} - \left(s\frac{r}{c_1} + 1\right) e^{-\frac{r}{c_1} s} \right] \right. \\
&\left. + \frac{\delta_{i(j+1)} n_{j+2} - \delta_{i(j+2)} n_{j+1}}{c_2 s} e^{-\frac{r}{c_2} s} \right\}.
\end{aligned} \tag{39}$$

## References


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